and the second integration, Eq. (6), yields

$$G(\infty) = 1 = Pr[1 + (2 - \beta)\gamma] \frac{m^2}{\pi a^2} \tan^{-1}m +$$

$$Pr(2 - \beta)\gamma \left\{ \frac{m}{2\pi a^2} - \frac{m^2}{4a^2} + \frac{1}{4a(\pi)^{1/2}} \left[m^3 - 1 + \frac{1}{(1+m^2)^{1/2}} \left(\frac{1 - m^5}{m} \right) \right] \right\} \quad (9)$$

Given β, a^2 may be determined from Ref. 1, Eq. (1.11), and it is

$$a^2 = \tfrac{1}{4} + \beta[\tfrac{1}{4} + (1/2\pi)] \quad (10)$$

For a given Prandtl number Pr and γ, m may be determined from Eq. (9).

Several numerical calculations have been performed, and they are tabulated in Table 1. Comparison between the approximate and Levy's[2] "exact" solutions is made. It may be noted that the results of Ref. 2 are negative but by a transformation may be made to conform to the results of this paper. The nondimensional temperature function of this paper is $\theta(\eta)$. Denoting the temperature function of Ref. 2 by $\theta_1(\eta)$, the relationship between the two functions is

$$\theta = 1 - \theta_1 \quad (11)$$

and, therefore, the derivatives are

$$d\theta/d\eta = -(d\theta_1/d\eta) \quad (12)$$

Although only one iteration was used, the results are quite satisfactory, especially for the case when $\beta = \gamma = 0$.

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Continued Comments on the Collapse of Pressure-Loaded Spherical Shells

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During recent years, continued theoretical and experimental interest has been expressed on the collapse of pressure-loaded spherical shells. However, to date there still are important unanswered questions concerning certain discrepancies between tests and the theories or semi-empirical plots. Such comparisons are shown in Figs. 2, 3, and 7 of Refs. 3, 5, and 6, respectively. The spread of data shown in Fig. 2 of Ref. 3 is so large that one wonders whether there really is any rational answer for this phenomenon. Fortunately, the author already has stated in Ref. 1 certain factors contributing to this scatter. Basically, the shell behavior is sensitive to initial irregularities that always exist in practice. The presence of these deviations overshadows any effect of the angle subtended by the shell segment; therefore, this angle is eliminated from the problem. Furthermore, the maximum compressive strain that the shell can sustain is considered as the true measure of the collapse strength.

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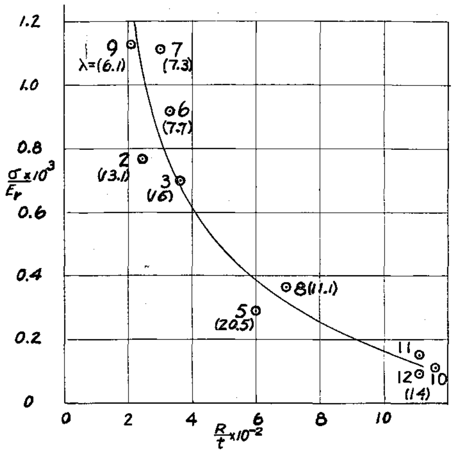

Fig. 1 Plot of data of Homewood, Brine, and Johnson

Based on these considerations, a plot of the test data given in Ref. 3 has been made and is shown in Fig. 1 of the present note. The numbers near the circles are the specimen numbers quoted in Ref. 3. The curve shown is approximately the lowest curve given in Fig. 1 of Ref. 1. The numbers in the parentheses are values of what is believed to be a significant parameter λ used in plotting data in Refs. 3, 5, and 6. It is seen that this parameter does not appear to correlate too well in the present plot. In general, the data presented here appear to be much more consistent and rational than when plotted vs λ.

An effort will be made to interpret the test data given in Refs. 4 and 7 as well as other known data. Then, design curves will be established in light of all these data.

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Transition Relations across Oblique Magnetohydrodynamic Shock Waves

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In magnetohydrodynamics, analogs of the usual gasdynamic shocks occur, and transition relations across hydromagnetic shock waves have been considered by de Hoffman and

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Teller,[1] Lüst,[2] Helfer,[3] Friedrichs,[4] Bazer and Ericson,[5] Kanwal,[6] Gundersen,[7] and others.

Although all flow parameters behind a conventional gas-dynamic shock wave may be determined by use of the Rankine-Hugoniot conditions if the flow in front of and one quantity behind the shock are specified, two parameters are required for a normal hydromagnetic shock wave, viz., one giving the shock strength and one giving a measure of the applied transverse field. All flow parameters behind the shock then may be expressed in terms of these and the known flow in front of the shock.[7]

It is the purpose of the present paper to show that all flow quantities behind an oblique hydromagnetic shock wave may be expressed in terms of the flow quantities in front of the shock and three parameters, viz., the shock strength, one giving a measure of the applied field and one giving a measure of the obliqueness of the applied field with respect to the shock front. Throughout, it is assumed that the fluid under consideration is an ideal, inviscid, non-heat-conducting gas of infinite electrical conductivity with constant specific heats.

Let the shock front be perpendicular to the x axis of an (x,y) coordinate system, and let $\mathbf{B} = (B_x, B_y, 0)$, $\mathbf{u} = (u_x, u_y, 0)$, c, ρ, U, $v = u_x - U$, $\mathbf{b} = (b_x, b_y, 0) = (B_x^2/\mu\rho, B_y^2/\mu\rho, 0)$, P, μ, and γ be the induction, particle velocity, local speed of sound, density, velocity of the shock front (assumed parallel to the x axis), flow velocity relative to the shock front, Alfvén speed, pressure, permeability, and ratio of specific heats. Let the regions in front of and behind the shock be denoted by the subscripts 1 and 2, respectively. Then the hydromagnetic analogs of the jump conditions across oblique shock fronts are[4]

$$B_{x1} = B_{x2} \tag{1}$$

$$\rho_1 v_1 = \rho_2 v_2 \tag{2}$$

$$P_1 + \rho_1 v_1^2 + B_{y1}^2/2\mu = P_2 + \rho_2 v_2^2 + B_{y2}^2/2\mu \tag{3}$$

$$\rho_1 v_1 u_{y1} - B_{x1} B_{y1}/\mu = \rho_2 v_2 u_{y2} - B_{x2} B_{y2}/\mu \tag{4}$$

$$v_1 B_{y1} - B_{x1} u_{y1} = v_2 B_{y2} - B_{x2} u_{y2} \tag{5}$$

$$v_1^2/2 - u_{y1}^2/2 + \gamma P_1/(\gamma - 1)\rho_1 + B_{y1}^2/\rho_1\mu = v_2^2/2 - u_{y2}^2/2 + \gamma P_2/(\gamma - 1)\rho_2 + B_{y2}^2/\rho_2\mu \tag{6}$$

For the case of no applied field, Eqs. (1–6) reduce exactly to the transition relations given by Courant and Friedrichs,[8] p. 299.

Introduce the parameters $\sigma = \rho_2/\rho_1$, $\tau = P_2/P_1$, $m_x = b_x/c$, $m_y = b_y/c$, $n_x = u_x/c$, $n_y = u_y/c$, and $B_y/B_x = \phi$. It will be shown that all flow quantities behind the shock may be expressed in terms of those in front and the three parameters σ, ϕ_1, and m_{y1}. From Eq. (2), it is immediately apparent that

$$\rho_2/\rho_1 = v_1/v_2 = \sigma \tag{7}$$

To simplify the derivation, a coordinate system is chosen such that the shock front is at rest and the flow velocity and magnetic field are parallel on both sides of the shock front.[1, 4] This is accomplished by choosing a coordinate system such that $\mathbf{u} = v\mathbf{B}/B_x$. Then Eq. (5) is satisfied identically, and, further, there is no distinction between u_x and v. Thus, it follows that

$$u_y/u_x = n_y/n_x = b_y/b_x = m_y/m_x = B_y/B_x = \phi \tag{8}$$

and

$$c_2^2/c_1^2 = \tau/\sigma \tag{9}$$

Then from the continuity of the normal component of the magnetic field, i.e., Eq. (1) and Eqs. (7) and (8), the following relations are derived:

$$B_{y2}/B_{y1} = \phi_2/\phi_1 \qquad b_{y2}^2/b_{y1}^2 = \phi_2^2/\sigma\phi_1^2 \tag{10}$$

$$b_{x2}^2/b_{x1}^2 = 1/\sigma \qquad m_{x2}^2/m_{x1}^2 = 1/\tau \tag{11}$$

$$m_{y2}^2/m_{y1}^2 = \phi_2^2/\tau\phi_1^2 \qquad n_{x2}^2/n_{x1}^2 = 1/\sigma\tau$$
$$n_{y2}^2/n_{y1}^2 = \phi_2^2/\sigma\tau\phi_1^2 \tag{12}$$

Since $u_x = v$ in the coordinate system used, Eq. (4) may be written as

$$\rho_1\phi_1[v_1^2 - b_{x1}^2] = \rho_2\phi_2[v_2^2 - b_{x2}^2]$$

or, solving for ϕ_2,

$$\phi_2 = \sigma\phi_1 \frac{(n_{x1}^2 - m_{x1}^2)}{(n_{x1}^2 - \sigma m_{x1}^2)} \tag{13}$$

which expresses ϕ_2 in terms of σ, ϕ_1, $m_{y1} = \phi_1 m_{x1}$, and the known flow in front of the shock.

Since Eqs. (1, 2, 4, and 5) have been used, only Eqs. (3) and (6) remain to be considered. These may be rewritten as

$$P_1 + \rho_1 v_1^2 + \rho_1 b_{y1}^2/2 = P_2 + \rho_2 v_2^2 + \rho_2 b_{y2}^2/2$$

$$v_1^2(1 - \phi_1^2)/2 + \gamma P_1/(\gamma - 1)\rho_1 + b_{y1}^2 = v_2^2(1 - \phi_2^2)/2 + \gamma P_2/(\gamma - 1)\rho_2 + b_{y2}^2$$

or, if all terms with subscript 2 are expressed in terms of those with subscript 1,

$$\rho_1 v_1^2(1 - 1/\sigma) + P_1\{1 - \tau + \gamma m_{y1}^2(1 - \phi_2^2/\phi_1^2)/2\} = 0 \tag{14}$$

$$\rho_1 v_1^2\{1 - \phi_1^2 + (\phi_2^2 - 1)/\sigma^2\} + P_1\{2\gamma(1 - \tau/\sigma)/(\gamma - 1) + 2\gamma m_{y1}^2(1 - \phi_2^2/\sigma\phi_1^2)\} = 0 \tag{15}$$

Since there exists a nontrivial solution to this linear system of homogeneous algebraic equations for P_1 and $\rho_1 v_1^2$, Eqs. (14) and (15), the determinant of the coefficients must vanish, which leads to the result

$$\tau = \frac{[\sigma^2(1 - \phi_1^2) + \phi_2^2 - 1][1 + \gamma m_{y1}^2(1 - \phi_2^2/\phi_1^2)/2] - 2\gamma\sigma(\sigma - 1)[1/(\gamma - 1) + m_{y1}^2(1 - \phi_2^2/\sigma\phi_1^2)]}{\sigma^2(1 - \phi_1^2) + \phi_2^2 - 1 - 2\gamma(\sigma - 1)/(\gamma - 1)} \tag{16}$$

which expresses τ in terms of ϕ_1 and m_{y1}. Because of Eqs. (13) and (16), Eqs. (7–12) express the flow quantities behind the shock in terms of those in front and the three parameters σ, ϕ_1, and m_{y1}. These results reduce to the previously presented[7] jump conditions across normal hydromagnetic shock waves in the limit of vanishing B_x. Some care must be exercised in carrying out this limit, e.g., it must be noted that

$$\lim_{B_x \to 0} (\phi_2/\phi_1) = \sigma$$

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